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Event-based Synchronization in Biology: Dynamics of Pulse Coupled Oscillators

Anton V. Proskurnikov and Ming Cao

Abstract—The principles of event-based control appear to be a backbone of many self-coordinating natural systems. Populations of flashing fireflies and cells of the cardiac pacemakers are believed to reach synchrony via event-based interactions, known as the pulse coupling. Synchronization via pulse coupling is also widely used in wireless sensor networks, allowing to avoid the packet exchange. In spite of serious attention to networks of coupled oscillators, there is a lack of results on their synchronization under general topology and phase-response curves. The most general, to the best of the authors' knowledge, result of this type (Wang et al., 2012) establishes synchronization of oscillators with a delay-advance phase-response curve under the assumption of strongly connected network topology. In this paper we relax the latter assumption to the existence of an oriented spanning tree, which is also necessary for the synchronization, being commonly adopted in multi-agent control.

Index Terms—Event-based control, oscillators, synchronization, pulse coupling

I. INTRODUCTION

As control systems become large scale and distributed over large distances, the costs of fast information processing in sensors, controllers, and actuators and of reliable communication between them, dramatically increase. This challenges one to develop control strategies that use communication and computational resources in a “parsimonious” way, giving birth to a new theory which integrates studies on control, communication and computing [1]. Whereas exact estimates of the minimal communication rate and control strategies providing this rate are quite complicated and mainly known for linear systems (see e.g. [2] and references therein), there are alternative approaches that, being non-optimal, are much simpler and still provide visible economy of communication resources. Among them is the paradigm of *event-based* (or *event-triggered*) control [3]–[6], proved to be especially efficient in control of large-scale networked and multi-agent systems [7]–[10].

Whereas the history of the event-based control in engineering is usually counted from the seminal papers [11], [12], long before that complex systems coordinating via event-based protocols had attracted much attention in biological and biophysical communities. One of the first phenomenological description of an event-based algorithm was given by J. Buck who studied synchronous flashes of a population of male fireflies in the dark: “...each community flash was initiated by a single flash, the others following almost instantaneously. Each individual apparently took his cue to flash from his more immediate neighbors, so that the mass flash took the form of a very rapid chain of overlapping flashes...” [13, p. 310]. In other words, a firefly interacts to its neighbors via reactions to the events of their flashing; an intensive light destroys this mechanism and the fireflies begin to flash asynchronously [13]. Analogous event-based interactions, referred to as the *pulse coupling*, lead to

synchronous hand claps of applauding audience [14] and maintain synchronous behavior in many biological networks, including cardiac and circadian pacemakers [15]–[17].

Mathematical models of the pulse-coupled oscillator (PCO) networks basically assume the trajectory of each oscillator to lie a small vicinity of a stable limit cycle. Confined to the oscillator motion along the cycle and neglecting the transversal dynamics, these models describe each oscillator with a scalar *phase variable* (varying on a unit circle or a segment of real line, at whose right endpoint the phase variable is reset to the left one). Unlike diffusively coupled oscillators (e.g. Kuramoto networks [18]), in PCO networks the duration of each interaction is negligibly small compared to the oscillator natural period. Passing some point on the cycle (corresponding to a fixed phase value), an oscillator fires an event (e.g. emits a pulse or some other stimulus), affecting some of its neighbors (the set of which is determined by the network topology). Upon receiving a stimulus from a neighbor, the oscillator trajectory is slightly disturbed, which is usually modeled as a *phase shift*, described by the *phase response (or resetting) curve* (PRC) [19].

Since the influential papers [20], [21], networks of PCO have been attracting considerable attention of system and control theorists, giving an instructive model of self-organizing under modest communication, inspired by nature. Pulse-coupling policies proved to be very efficient in the problem of wireless sensor network synchronization [22]–[25], as they save the communication resources and increase the network reliability by avoiding message exchange.

At the same time, there is the lack of mathematical results, ensuring synchronization in networks of PCOs with general PRCs and network topologies. Assuming the coupling strengths to be very weak, one can approximate such a network with a continuous Kuramoto-like model [23], [26], [27] that is now well studied (see e.g. a recent survey [28] and references therein). Without the “weak coupling” assumption, the dynamics of PCO networks is hybrid and has been examined mainly for the cases of all-to-all and ring coupling [20], [29]–[32]. One of the most general results on PCO synchronization has been obtained in [24] and establishes a synchronization in ensembles of identical PCO, provided their PRC is of the *delay-advance* type and the maximal deviation between their initial phases is less than half of the oscillation period. The criterion from [24] restricts the network to be *strongly connected*.

In the present paper, we extend the result of [24] to a wider class of networks, whose interaction graph is not necessarily strongly connected but is *rooted* or, equivalently, contains an *oriented spanning tree*. This is a standard assumption in the literature on consensus and synchronization [33], [34] and the minimal requirement under which the synchronization is possible. Our criterion, in particular, is applicable to a network with one leader and several followers, uncovered by the results of [24].

II. THE PROBLEM SETUP

Henceforth we deal with the following model of PCO networks, elaborated in [24], [25], [30]–[32]. Consider an ensemble of $N > 1$ identical oscillators, whose phase functions are $\theta_1(t), \dots, \theta_N(t) \in$

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$[0; 2\pi)$. The “free-run” dynamics of each oscillator is

$$\dot{\theta}_i(t) = \omega, \quad \theta_i(t-) = 2\pi \Rightarrow \theta_i(t+) = 0. \quad (1)$$

Here the frequency $\omega > 0$ is constant and, as usual, we assume that passing through the value 2π , the phase is reset to 0.

In order to synchronize their phase variables, the oscillators apply the following event-based control protocol. As the phase of an oscillator passes through some fixed value (common for all oscillators), it fires an *event*, e.g. by sending out a stimulus. Without loss of generality, we assume that the j th oscillator fires at time $t_* > 0$ if $\theta_j(t_*) = 2\pi$. At this moment, its phase is reset to 0, whereas the remaining oscillators may acquire phase shifts:

$$\theta_i(t_*) = \theta_i(t_*-) + c a_{ij} \Phi(\theta_i(t_*-)) \mod 2\pi \quad \forall i \neq j. \quad (2)$$

Here the values $a_{ij} \in \{0; 1\}$ determine the *interaction topology* of the network: the j th oscillator affects the i th one if and only if $a_{ij} = 1$. The number $c > 0$ stands for the *coupling strength* in the network and the map $\Phi : [0; 2\pi] \rightarrow \mathbb{R}$ is referred to as the *phase response curve* (PRC). We always assume that Φ is continuous and $\Phi(0) = 0 = \Phi(2\pi)$, which means that a firing oscillator affects none of those oscillators, that are synchronous to it: if $\theta_i(t_*) = 2\pi = \theta_j(t_*)$, then $\theta_i(t_*) = \theta_j(t_*) = 0$.

In order to make the model just described complete, it remains to define the behavior when several oscillators fire simultaneously (a “cluster burst” [31] occurs). We follow the model from [31] and consider the resulting phase jump as the (nonlinear) *superposition* of several jumps. Let the i th oscillator be affected at time $t = t_*$ by k simultaneously firing oscillators, then its new phase is

$$\theta_i(t_*) = \underbrace{\Psi \circ \Psi \circ \dots \circ \Psi}_{k \text{ times}}(\theta_i(t_*-)), \quad \Psi(\theta) \triangleq \theta + c\Phi(\theta) \mod 2\pi. \quad (3)$$

The mapping $\Psi(\theta)$ is sometimes referred to as the *phase transition curve* (PTC). If the i th agent at time t_* has the phase $\theta_i(t_*) = \theta$ and is affected by the event of some other oscillator, its new phase jumps to $\theta_i(t_*) = \Psi(\theta)$ due to (2). Equation (3) states that the burst of k simultaneous events result in the sequence of k such instantaneous jumps.

The definition (3) implies that trajectories continuously depend on the initial conditions, unlike the models with the *additive* effect of several events [21], [30]), assuming that for any i one has

$$\theta_i(t_*) = \theta_i(t_*-) + c \sum_{j: \theta_j(t_*) = 2\pi} a_{ij} \Phi(\theta_i(t_*-)) \mod 2\pi. \quad (4)$$

Protocols (4) are usually examined either for weak couplings $c \ll 1$ by reducing them to Kuramoto models [21], [23], [27] or under the assumption that oscillators’ phases are non-overlapping $\theta_i(t) \neq \theta_k(t)$, which is valid e.g. for almost all trajectories under all-to-all interactions [20], [30]. Another definition, which allows the phase transition map under several events to be multi-valued, was proposed in the very recent paper [32], which also deals only with special interaction graphs.

Our goal is to find out conditions which guarantee asymptotic synchronization in the networked system (1),(4).

Definition 1: The phases $\theta_i(\cdot)$ ($i = 1, \dots, N$) synchronize if

$$e^{i\theta_i(t)} - e^{i\theta_j(t)} \xrightarrow[t \rightarrow \infty]{} 0.$$

Here, as usual, $e^{i\varphi} = \cos \varphi + i \sin \varphi$ for any $\varphi \in \mathbb{R}$.

It is commonly known that even under simple undirected topologies synchronization in general is not possible for some initial conditions [20], [30]. The simplest example is a pair ($N = 2$)

of coupled oscillators, where a_{12}, a_{21} may be arbitrary and PRC is such a map that $\Phi(0) = \Phi(\pi) = \Phi(2\pi) = 0$. Consider a solution, starting at $\theta_1(0) = 0$ and $\theta_2(0) = \pi$. The first event is fired by the second oscillator at time $t_2^1 = \pi/\omega$. One has $\theta_1(t_2^1-) = \pi = \theta_1(t_2^1+)$ and hence the next event is fired by the first oscillator at time $t_1^1 = 2\pi/\omega$; after this $\theta_1(t_1^1+) = 0$, $\theta_2(t_1^1+) = \pi$, that is, the initial configuration is recovered.

In the next section a criterion will be given, which guarantees synchronization for solutions, whose initial phases $\theta_i(0)$ differ by less than π . Synchronization under this condition is well known for Kuramoto models [28] and was proved in [24] for networks of PCO with delay-advance PRC under the assumption of strong connectivity of the network, which will be relaxed in the next section.

III. MAIN RESULTS

In this section, we establish a criterion of synchronization in the network (1),(2),(3) under several assumptions. The first condition requires that the oscillators have a *delay-advance* PRC [24]. Examples of such PRC may be found e.g. in [24], [30].

Assumption 1: The function $\Phi : [0; 2\pi] \rightarrow \mathbb{R}$ is continuous with $\Phi(0) = \Phi(\pi) = \Phi(2\pi) = 0$. Furthermore, $\Phi(x) < 0$ if $x \in (0; \pi)$ and $\Phi(x) > 0$ when $x \in (\pi; 2\pi)$.

Assumption 1 implies that during event-triggered interaction (2) the phase of the influenced oscillator $\theta_i(t)$ jumps towards that of the firing oscillator $\theta_j(t)$ (modulo 2π) except for the case where $\theta_i(t_*) = 0$. If $\theta_i(t_*) \in (0; \pi)$, the i th oscillator is ahead of the firing one and is to be delayed ($\Phi(\theta_j(t_*-)) < 0$), whereas for $\theta_i(t_*) \in (\pi; 2\pi)$ it has to be advanced. Our next assumption prohibits overshoots during such a synchronization.

Assumption 2: The coupling strength c is chosen in a way that

$$\Psi(x) > 0 \quad \forall x \in (0; \pi), \quad \Psi(x) < 2\pi \quad \forall x \in (\pi; 2\pi), \quad (5)$$

where $\Psi(x) = x + c\Phi(x)$ stands for the phase transition map.

Assumption 2 is always satisfied with sufficiently small $c > 0$ if

$$\inf_{x \in (0; \pi)} \frac{\Phi(x)}{x} > -\infty, \quad \sup_{x \in (\pi; 2\pi)} \frac{\Phi(x)}{2\pi - x} < \infty.$$

Analogous conditions, providing that oscillators do not overrun, are often adopted in the literature, see e.g. [31]. Some restriction, imposed by Assumptions 1 and 2, is the impossibility of synchronization in finite time, proved in [24]. On the other hand, such a convergence requires the PRC to be linear in the vicinity of 0 and 2π [24], which is a basic case in the theory of wireless sensor networks but not for biological oscillators [30]. An important implication of these assumptions is the absence of the Zeno behavior: it will be shown that the instants of consequent events can have no accumulation point. Moreover, two events fired by the *same* oscillator are always separated by a positive dwell-time.

Lemma 1: Let Assumptions 1 and 2 hold and $T = 2\pi/\omega$ be the natural period of oscillator (1). If the j th oscillator fires an event at time $t_* > 0$, then $0 < \theta_i(t) \leq \omega(t - t_*) < \pi$ as $t \in (t_*; t_* + T/2)$. In particular, it fires no events as $t \in (t_*; t_* + T/2)$.

On the other hand, the time between two consequent events is estimated from above by T , as implied by the following lemma.

Lemma 2: Let Assumptions 1 and 2 hold and $\theta_j(t_0) \in [\pi; 2\pi)$. Then the j th oscillator fires (only) once during the interval $(t_0; t_0 + T/2]$. Moreover, $\theta_j(t) - \theta_j(t_0) \geq \omega(t - t_0)$ as $t \in (t_0; t_1)$, where t_1 is the time of event.

At any time $t_0 \geq 0$ either one of the phases $\theta_i(t_0)$ belongs to $[\pi; 2\pi)$ and hence the next event will be fired no later than in $T/2$ seconds, or otherwise $\theta_j(t) \in [0; \pi)$ and the phases evolve in

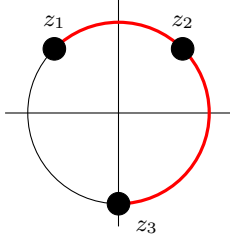


Fig. 1: $d(z_1, z_2, z_3) = 5\pi/4$

accordance with (1) till one of them reaches 2π . This obviously occurs at the instant $t = t_0 + (2\pi - \max_j \theta_j(t_0))/\omega \leq T$. In both situations one can see the interim between two consequent events in the networks is not greater than T .

As was shown in Section II, synchronization under arbitrary initial conditions cannot be proved since $\Phi(\pi) = 0$. There exist two-cluster solutions, where the clusters' phases deviate by π ; moreover, such solutions may be stable [31]. However, under Assumptions 1 and 2 such clusters cannot arise if the initial phases are "approximately" synchronized, deviating by less than π : $\max_{i,j} |\theta_j(0) - \theta_i(0)| < \pi$. We prove an even more general result, where the deviations are calculated modulo 2π .

We start with some preliminaries and notation. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ stand for the unit circle, whose points $z \in \mathbb{S}^1$ are in one-to-one correspondence $\theta \mapsto z = e^{i\theta}$ with phases $\theta = [0; 2\pi)$.

Definition 2: We refer to a closed connected subset of \mathbb{S}^1 (that is, a set $\{e^{i\omega} : \omega \in [a; b] \subset \mathbb{R}\}$) as an *arc* in \mathbb{S}^1 . Given an arc $L \subseteq \mathbb{S}^1$, let $|L| \in [0; 2\pi]$ stands for its length (in radians). For a finite sequence $\bar{z} = \{z_i\}_{i=1}^N \subset \mathbb{S}^1$ let $d(\bar{z}) \triangleq \min_{z_i \in L \forall i} |L|$ (the minimum is taken over all arcs, containing z_i). Analogously, for a sequence $\bar{\theta} = \{\theta_i\}_{i=1}^N \subset [0; 2\pi)$ let $d(\bar{\theta}) \triangleq d(e^{i\theta_1}, \dots, e^{i\theta_N})$.

For instance, for three points on Fig. 1 one has $d(z_1, z_2, z_3) = 5\pi/4$, one of the two minimal arcs is drawn in red.

Assumption 3: One has $d(\theta_1(0), \dots, \theta_N(0)) < \pi$, i.e. the minimal arc L_0 containing points $e^{i\theta_i(0)}$ has length $|L_0| < \pi$ (such an arc is obviously unique).

The following lemma shows that the condition from Assumption 3 remains valid at any time $t \geq 0$.

Lemma 3: Let Assumptions 1-3 hold and L_0 be the minimal arc from Assumption 3. Then for any $t \geq 0$ and $j = 1, \dots, N$ one has $e^{i\theta_j(t)} \in L_t \triangleq e^{i\omega t} L_0$. Moreover, the function $d(t) \triangleq d(\theta_1(t), \dots, \theta_N(t))$ is non-increasing and hence $d(t) \leq d(0) < \pi$.

Lemma 3 implies, in particular, that under Assumptions 1-3 the limit exists $d_* = \lim_{t \rightarrow \infty} d(\theta_1(t), \dots, \theta_N(t))$. The oscillators asymptotically synchronize, obviously, if and only if $d_* = 0$. An elegant corollary, which follows from this lemma, shows that under the aforementioned assumptions the sequence of events appears to be very regular.

Corollary 1: Under Assumptions 1-3 the claims are valid:

- 1) during the natural period $T = 2\pi/\omega$ any oscillator fires at least once and no more than two times;
- 2) between two events, fired by the same oscillators, all other oscillators fire (at least once).

Indeed, consider arbitrary initial phases $\theta_i(0) \in [0; 2\pi)$, satisfying Assumption 3. Let $\theta_* = \max_j \theta_j(0)$ and $J = \{j : \theta_j(0) = \theta_*\}$. The oscillators with indices from J fire first at time instant $t_* = T - \theta_*/\omega$, and $\theta_i(t_* -) \geq 2\pi - d(0) > \pi$ for any i . Therefore, all the remaining oscillators $i \notin J$ fire during $(t_*, t_* + d(0)/\omega]$ due to Lemma 2. Since $\theta_* \geq d(0) = \theta_* - \min_j \theta_j(0)$, all the oscillators

fire (for the first time) no later than $t_* + d(0)/\omega = T - (\theta_* - d(0))/\omega \leq T$. The next firing time for the oscillator with index $j \in J$ is greater than $t_* + T/2 > t_* + d(0)/2$, so these "leading" oscillators fire for the second time *after* the other oscillators have fired. Notice, however, that in general $t_* + T/2 < T$, so one or several oscillators from J can fire twice during the period T . Finally, for any t one has $d(t) = d(\theta_1(t), \dots, \theta_N(t)) \leq d(0) < \pi$, so the arguments may be retraced, replacing $[0; T]$ with $[t; t + T]$ and $\theta_i(0)$ with $\theta_i(t)$.

We are now ready to formulate our main result, stating that the synchronization takes place under the mentioned assumptions, provided that the network topology has an *oriented spanning tree*. Consider a graph $G = (V, E)$, where $V = \{1, \dots, N\}$ is the set of nodes, corresponding to oscillators, and E is the set of arcs, defined by the matrix $A: E = \{(i, j) : a_{ji} = 1\}$. Notice that, as usual in multi-agent control [33], the direction of an arc corresponds to the influence (or information flow): the j th oscillator affects the i th, i.e. $a_{ij} = 1$, if and only if an arc (j, i) from j to i exists.

Theorem 1: Let Assumptions 1-3 hold and the graph G have an oriented spanning tree. Then the oscillators synchronize.

It should be noticed that the assumption on the existence of an oriented spanning tree is commonly adopted in multi-agent control [33], [34], being the weakest condition under which the synchronization can be proved. If this condition fails, there exist two disjoint sets of nodes $V_1, V_2 \neq \emptyset$, such that none of them has incoming arcs [35, Theorem 5]. In particular, two groups of oscillators, corresponding to the nodes from V_1 and those from V_2 , communicate neither to each other nor to the remaining network, so synchronization between them is not possible.

Proofs of Lemmas 1-3 and Theorem 1 are given in Section V.

IV. NUMERICAL SIMULATION

In this section, the result of Theorem 1 is confirmed by numerical tests. We simulate a network of $N = 4$ identical oscillators with the natural frequency $\omega = 1$ rad/s, whose interaction topology is shown in Fig. 2. Notice that the graph in Fig. 2 has an oriented

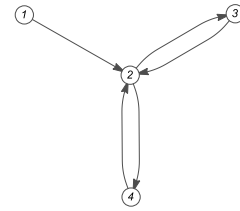


Fig. 2: The network topology

spanning tree, but is *not* strongly connected because the phase of the "leading" oscillator 1 is unaffected by the others.

The oscillators start with phases $\theta_1 = \pi/2$, $\theta_2 = 0.3\pi$, $\theta_3 = 0.03\pi$ and $\theta_4 = 0.9\pi$ (red).

First we simulate the dynamics of network under the delay-advanced PRC $\Phi(\theta) = -\sin \theta$ (see Fig. 3a) and the coupling strength $c = 0.4$. The dynamics of oscillators' phases θ_1 (blue), θ_2 (orange), θ_3 (green) and θ_4 (red) are shown on Figs. 4 and 5. The diagram of events is displayed in Fig. 6: the point (t, i) on the plot in Fig. 6 (where $t \geq 0$ and $i \in \{1, 2, 3, 4\}$) indicates that the i th oscillator fires an event at time t .

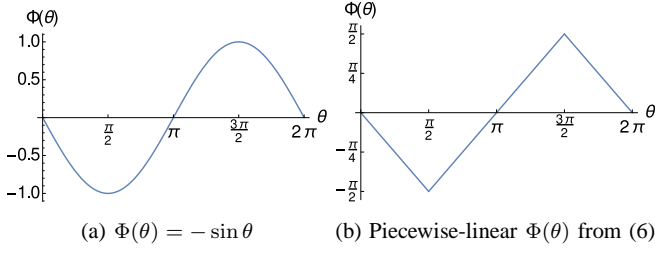


Fig. 3: Two PRC maps

Our second test deals with piecewise-linear delay-advance PRC

$$\Phi(\theta) = \begin{cases} -\theta, & \theta \in [0; \pi/2] \\ \theta - \pi, & \theta \in [\pi/2; 3\pi/2] \\ 2\pi - \theta, & \theta \in (3\pi/2; 2\pi], \end{cases} \quad (6)$$

depicted in Fig. 3b, the coupling strength being $c = 0.5$. The dynamics of oscillators' phases $\theta_i(t)$ are shown in Figs. 7 and 8; the events diagram is displayed in Fig. 9.

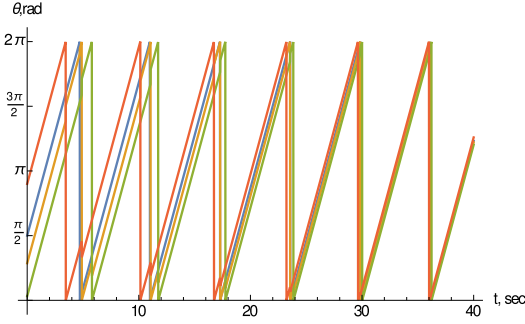


Fig. 4: Oscillator phases, $\Phi(\theta) = -\sin \theta$, $c = 0.4$.

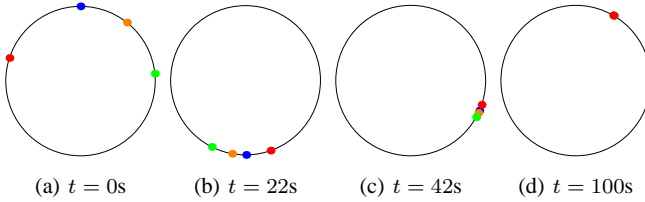


Fig. 5: Oscillator phases on S^1 , $\Phi(\theta) = -\sin \theta$, $c = 0.4$.

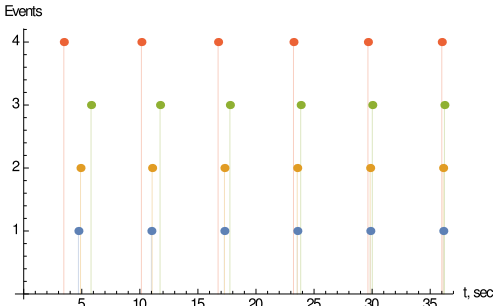


Fig. 6: Event diagram, $\Phi(\theta) = -\sin \theta$, $c = 0.4$.

Both numerical examples confirm that the oscillators get synchronous under protocol (2), as claimed by Theorem 1.

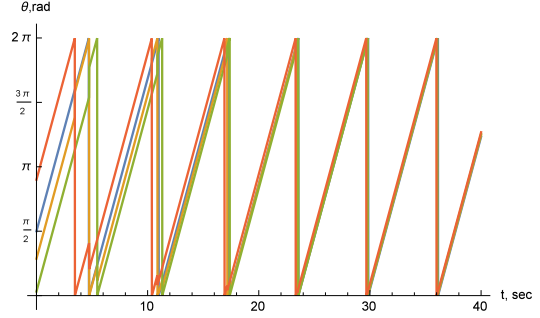


Fig. 7: Oscillator phases, $\Phi(\theta)$ from (6), $c = 0.5$.

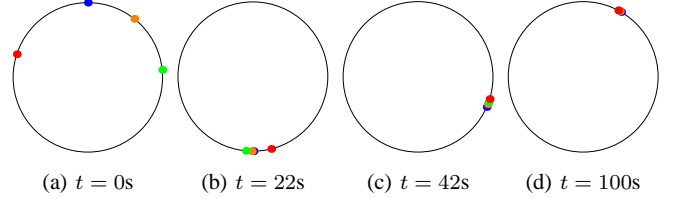


Fig. 8: Oscillator phases on S^1 , $\Phi(\theta)$ from (6), $c = 0.5$.

V. PROOFS

This section is organized as follows. We start with proving of Lemmas 1 and Lemma 2, which establish important properties of the event sequence and entail Lemma 3. Then we pass to the proof of Theorem 1, based on the construction of the *return map*.

Proof of Lemma 1. For a short while after an event (or burst of events) occurs no other events can be fired and the phases evolve in accordance with (1). So $0 < \theta_i(t) \leq \omega(t - t_*)$ as $t \in [t_*; t_* + \varepsilon)$ if $\varepsilon > 0$ is small (and, in fact, the equality holds $\theta_i(t) = \omega(t - t_*)$). Let ε_0 be the maximal ε with such a property. Our goal is to show that $\varepsilon_0 \geq T/2$. Assume on the contrary that $\varepsilon_0 < T/2$. Since ε_0 is maximal, at the time instant $t' = t_* + \varepsilon_0$ the phase θ_i is affected by single or multiple events. Since $0 < \theta_i(t' -) < \pi$, one has $0 < \theta_i(t'_+) < \theta_i(t' -)$ due to Assumption 1 and 2. After this, no other events are fired for a while and hence $0 < \theta_i(t) \leq \omega(t - t_*)$ also for some interval $t \in [t'; t' + \varepsilon)$. One arrives at the contradiction with the maximality of ε_0 , thus proving the lemma. ■

Proof of Lemma 2 is based on the same idea. For small $\varepsilon > 0$ we have no events during the interval $(t_0; t_0 + \varepsilon)$ and hence $\theta_i(t) \geq \omega(t - t_*)$ as $t \in [t_0; t_0 + \varepsilon)$. Let ε_0 be the maximal ε with such a property. Our goal is to show that at time $t_1 = t_0 + \varepsilon_0$ the j th oscillator fires an event, so that $\theta_j(t_1 -) = 2\pi$. Indeed, since ε_0 is maximal, at the time instant t_1 single or multiple events are fired. If the j th oscillator does not fire, its phase either remains unchanged or is affected by some events and hence $\theta_j(t_1 -) < \theta_j(t_1 +) < 2\pi$ due to Assumptions 1 and 2. After this, no other events are fired for a while and hence $0 < \theta_i(t) \geq \omega(t - t_*)$ also for some interval $t \in [t_1; t_1 + \varepsilon)$ which contradicts to the maximality of ε_0 . Therefore, the j th oscillator fires at time t_1 and $\omega(t_1 - t_0) \leq 2\pi - \theta_j(t_0)$, which proves the lemma. ■

We are now ready to prove Lemma 3.

A. Proof of Lemma 3

Let the sequence of events in the system be triggered at moments $0 < \tau_1 < \tau_2 < \dots$; Lemma 1 implies that any bounded interval $[0; t]$ contains only finite number of τ_i . By definition, for $t < \tau_1$ and $t \in (\tau_i; \tau_{i+1})$ equation (1) holds, so $e^{i\theta_i(t)} = e^{i\omega t + i\theta_i(0)} \in L_t$ for $t < \tau_1$. Therefore, $e^{i\theta_i(t_1 -)} \in L_{t_1} \forall i$ by continuity.

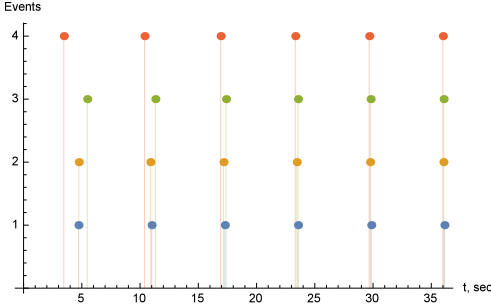


Fig. 9: Event diagram, $\Phi(\theta)$ from (6), $c = 0.5$.

Our goal is to show that $e^{i\theta_i(t_1+)} \in L_{t_1} \forall i$. Indeed, let the j th oscillator fires an event so that $\theta_j(t_1-) = 2\pi$, and the i th one is affected, choosing its phase in accordance with (2). Note that $\theta_i(t_1-) \neq \pi$, since $|L_{t_1}| < \pi$ by Assumption 3 and L_{t_1} contains both phases $\theta_j(t_1-) = 2\pi$ and $\theta_i(t_1-)$. Assume that $\theta_i(t_1-) \in (0; \pi)$. Then for any $\theta \in [0; \theta_i(t_1-)]$ one has $e^{i\theta} \in L_{t_1}$ as $|L_{t_1}| < \pi$ (see Fig.10). This entails that $e^{i\theta_i(t_1+)} \in L_{t_1}$ due to Assumptions 1,2. In the case of multiple events, one also has $e^{i\theta_i(t_1+)} \in L_{t_1}$ due to Assumptions 1,2 and (3). The case where $\theta_i(t_1-) \in (\pi; 2\pi)$ is considered analogously (see Fig.10). Thus we have $e^{i\theta_i(t_1+)} \in L_{t_1} \forall i$ so that $e^{i\theta_i(t)} \in L_t$ for $t < \tau_2$. Retracing the same arguments, one proves that $e^{i\theta_i(t)} \in L_t$ for $t < \tau_3$ and so on, proving thus the first claim of Lemma 3 and showing that $d(t) = d(\theta_1(t), \dots, \theta_N(t)) \leq d(\theta_1(0), \dots, \theta_N(0))$. By substitution $\theta_i(0) \mapsto \theta_i(s)$ one shows that $d(t+s) \leq d(s)$ for any $t, s \geq 0$, i.e. d is non-increasing. ■

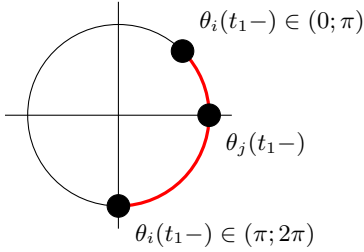


Fig. 10: Illustration to the proof of Lemma 3

B. The return map and its continuity

In this subsection, we introduce some auxiliary constructions to be used in the proof of Theorem 1. The main of them is the *return map* [20], [31], which describes the evolution of the initial state during one full “cycle” of oscillator firing.

For a vector $\bar{\xi} = (\xi_1, \dots, \xi_N) \in [0; 2\pi]^N$ such that $d(\bar{\xi}) < \pi$ consider the solution of our system $\bar{\theta}(t) = (\theta_1(t), \dots, \theta_N(t))$ with initial conditions $\theta_i(0) = \xi_i \forall i$. Note that we extend here the notion of solution, allowing that $\theta_i(0) = 2\pi$ for some i . In this situation, we assume that the corresponding oscillators immediately fire events and reset their phases to 0, after which the system evolves as usual. Let $t_*^i(\bar{\xi})$ stand for the time when the i th oscillator triggers its first event, Corollary 1 implies that $0 \leq t_*^j(\bar{\xi}) \leq T$. Let $t_*(\bar{\xi}) = \max_j t_*^j(\bar{\xi})$ be the time when the last of oscillators has just fired, the system state after this event is $\bar{\theta}_* \triangleq \bar{\theta}(t_*(\bar{\xi})+)$.

Definition 3: We call the map $S : \bar{\xi} \mapsto \bar{\theta}_*(\bar{\xi})$ the *return map* of the system.

The following lemma, showing that S is continuous, is a cornerstone tool in proving the synchronization.

Lemma 4: Let $\bar{\xi}^n \in [0; 2\pi]^N$ be a sequence that has a limit $\bar{\xi} = \lim_{n \rightarrow \infty} \bar{\xi}^n$. Then $t_*^j(\bar{\xi}^n) \rightarrow t_*^j(\bar{\xi})$ for all $j = 1, 2, \dots, N$ and $S(\bar{\xi}^n) \rightarrow S(\bar{\xi})$, i.e. S continuous at any point of its definition.

Proof: Suppose first for simplicity that $t_*^i(\bar{\xi}) \neq t_*^j(\bar{\xi})$ for all $i \neq j$. Renumbering the oscillators, without loss of generality one may assume that $t_*^1(\bar{\xi}) < \dots < t_*^N(\bar{\xi})$, i.e. $\xi_1 = \max_j \xi_j$. Then we have $t_*^1(\bar{\xi}) = T - \xi_1/\omega$ and thus it is obvious that $t_*^1(\bar{\xi}^n) = T - \xi_1^n/\omega \rightarrow t_*^1(\bar{\xi})$ since $\xi_1^n = \max_j \xi_j^n$ as n is sufficiently large. Furthermore, it is obvious that the system state after the first event $\bar{\xi}_+^n \triangleq \theta(t_*^1(\bar{\xi}^n)+) \rightarrow \bar{\xi}_+ \triangleq \theta(t_*^1(\bar{\xi})+)$ as $n \rightarrow \infty$. Substituting now $\bar{\xi} \mapsto \bar{\xi}_+$ and renumbering oscillators $(1, 2, \dots, N) \mapsto (2, 3, \dots, N, 1)$, one can obtain now that $t_*^2(\bar{\xi}^n) = t_*^1(\bar{\xi}^n) + t_*^1(\bar{\xi}_+^n) \rightarrow t_*^1(\bar{\xi}) + t_*^1(\bar{\xi}_+) = t_*^2(\bar{\xi})$ and $\theta(t_*^2(\bar{\xi}^n)+) \rightarrow \theta(t_*^2(\bar{\xi})+)$ and so on, getting finally that $\theta(t_*^N(\bar{\xi}^n)+) \rightarrow \theta(t_*^N(\bar{\xi})+)$ and $S(\bar{\xi}^n) = \theta(t_*^N(\bar{\xi}^n)+) \rightarrow S(\bar{\xi}) = \theta(t_*^N(\bar{\xi})+)$ as $n \rightarrow \infty$.

The case where some oscillators may fire simultaneously is considered likewise but for one difference. Assume again that oscillators are sorted in the order of their first firing $t_*^1(\bar{\xi}) \leq \dots \leq t_*^N(\bar{\xi})$. Suppose e.g. the $t_*^l(\bar{\xi}) = t_*^2(\bar{\xi}) = \dots = t_*^l(\bar{\xi}) < t_*^{l+1}(\bar{\xi})$, that is, the first l oscillators fire simultaneously. It is easily shown then that $t_*^i(\bar{\xi}^n) \rightarrow t_*^i(\bar{\xi})$ for $i = 1, 2, \dots, l$ as $n \rightarrow \infty$, and for large n the first l oscillators fire earlier than the remaining ones. A crucial difference is that one can no longer prove that $\theta(t_*^1(\bar{\xi}^n)+) \rightarrow \theta(t_*^1(\bar{\xi})+)$ since the state in the right hand side was produced by the cluster burst of l events, that may be not simultaneous in the n th system. One can show, however, that

$$\bar{\xi}_+^n \triangleq \theta(\max_{1 \leq i \leq l} t_*^i(\bar{\xi}^n)+) \rightarrow \bar{\xi}_+ \triangleq \theta(t_*^1(\bar{\xi})+).$$

Notice that the value in the left-hand state is the state after all l events have fired. Substituting now $\bar{\xi} \mapsto \bar{\xi}_+$ and renumbering oscillators, one again can iterate this procedure as was done above in the non-degenerate case, proving the claim of Lemma. ■

Lemma 3 implies that S is a non-expansive map in the sense of the “diameter” d : $d(S(\bar{\xi})) \leq d(\bar{\xi})$ whenever $d(\bar{\xi}) < \pi$. The following lemma shows that if the network topology has an oriented spanning tree, then the iteration $S^{N-1} = S \circ S \circ \dots \circ S$ is featured by much stronger contractivity property.

Lemma 5: Let the graph G defined in Section III have an oriented spanning tree. Then if $0 < d(\bar{\xi}) < \pi$, then $d(S^{N-1}(\bar{\xi})) \leq d(\bar{\xi})$.

Proof: Let $L(t)$ be the arc of minimal length, containing the points $e^{i\omega_j(t)}$ and $J_-(t)$, $J_+(t)$ stand for the sets of indices of oscillators, being its endpoints at time t (the shortest turn from the points from J_- to those from J_+ is counterclockwise). A closer analysis of the proof of Lemma 3 reveals that at the time of event t_* the following alternatives are possible:

- oscillators from $J_-(t_*-), J_+(t_*-)$ are not affected by the firing oscillator(s), then $|L(t_*+)| < |L(t_*-)|$ and $J_-(t_*+) = J_-(t_*-)$, $J_+(t_*+) = J_+(t_*-)$;
- some of the “extremal” oscillators are affected, but $|L(t_*+)| = |L(t_*-)|$. In this case $J_-(t_*+) \subseteq J_-(t_*-)$ and $J_+(t_*+) \subseteq J_+(t_*-)$ and at least one inclusion is strict;
- the length is decreasing: $|L(t_*+)| < |L(t_*-)|$.

It may be noticed that during the cycle of N consequent events (triggered by all of oscillators) at least once the second or third alternative must take place. Otherwise, one would have two disjoint sets of nodes $J_-(t_*-), J_+(t_*-)$ without incoming arcs that is not possible for a graph with oriented spanning tree [35, Theorem 5]. Since the cardinality of $J_-(t) \cup J_+(t)$ is not greater than N , after $N - 1$ cycles the length of arc necessarily decreases. ■

C. Proof of Theorem 1

Henceforth the assumptions of Theorem 1 are supposed to hold and in Assumption 3 one has $d(0) = d_0 < \pi$. Let $\bar{\theta}(t) = (\theta_1(t), \dots, \theta_N(t))$ be a solution. For synchronization it is sufficient to prove that $d(t) \rightarrow 0$ as $t \rightarrow \infty$ which, in its turn, is entailed by $d(S^n(\bar{\theta}(0))) \rightarrow 0$ as $n \rightarrow \infty$. Indeed, from Corollary 1 and Lemma 3 it follows that $d(S(\bar{\theta}(0))) \geq d(\bar{\theta}(T))$ and, therefore, $d(S^n(\bar{\theta}(0))) \geq d(\bar{\theta}(nT))$ for any $n = 1, 2, \dots$

Consider a compact set $K = \{\bar{\xi} \in [0; 2\pi]^N : d(\bar{\xi}) \leq d_0\}$. We are going to show that any sequence $d(S^n \bar{\xi}) \rightarrow 0$ as $n \rightarrow \infty$ for any $\bar{\xi} \in K$. Suppose, on the contrary, that $d_0 = \lim_{n \rightarrow \infty} d(S^n \bar{\xi}) > 0$. Since K is compact, there exists a sequence $n_k \rightarrow \infty$ such that $S^{n_k} \bar{\xi} \rightarrow \bar{\xi}_0 \in K$ and $d_0 = d(\bar{\xi}_0)$. Therefore, $S^{N-1+n_k} \bar{\xi} \rightarrow S^{N-1} \bar{\xi}_0$ and hence $d_0 = d(S^{N-1} \bar{\xi}_0)$. One arrives at the contradiction with Lemma 5. ■

VI. CONCLUSIONS

In this paper, we examine the protocol for oscillator synchronization, based on pulse-coupled interactions. We prove that the oscillators get synchronized for a general pulse response curve of the delay-advance type, provided that the topology of the network has an oriented spanning tree and the maximal distance between the initial phases is less than π . The extensions for more general classes of networks, including those with positive *refraction periods* [24], [25] and time-varying topology, is the subject of ongoing research.

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